DOCUMENT RESUME

ED 331 886 TM 016 446

AUTHOR Muraki, Eiji
TITLE Multiple-Group Full-Information Factor Analysis.
PUB DATE Apr 91
NOTE 17p.; Paper presented at the Annual Meeting of the American Educational Research Association (Chicago, IL, April 3-7, 1991).
PUB TYPE Reports - Evaluative/Feasibility (142) -- Speeches/Conference Papers (150)

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Data Analysis; Difficulty Level; *Equations (Mathematics); *Estimation (Mathematics); *Factor Analysis; Groups; Guessing (Tests); *Mathematical Models; Simulation; *Test Items
IDENTIFIERS Confirmatory Factor Analysis; *Full Information Factor Analysis; *Multiple Group Factor Analysis; TESTFACT

ABSTRACT

Multiple group factor analysis is described and illustrated through a simulation involving 5,000 examinees. The estimation process of the group factors were implemented using the TESTFACT program of Wilson and others (1987). Group factor analysis is described as a special case of confirmatory factor analysis. Group factors can be computed based on the collapsed gradient and Hessian matrix by summing up the elements according to the item group structure. This full information method can control guessing and the difficulty of each item. Implications of the simulation for additions to the TESTFACT program are discussed. Confirmatory factor analyses, including the group level factor structure, are considered appropriate processes for analyzing test data. Nine tables present data from the simulation study. (SLD)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED331886

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☒ This document has been reproduced as received from the person or organization originating it.

☐ Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY

EIJI MURAKI

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

Multiple-Group Full-Information Factor Analysis

Eiji Muraki

Statistical and Psychometric Research and Services
Educational Testing Services
Princeton, New Jersey 08541
609-734-1644

Paper presented at the annual meeting of
the American Educational Research Association
Chicago, Illinois
April 3-5, 1991

TM016446

BEST COPY AVAILABLE

In the multiple-group factor analysis, the factor is defined as the sum of the scores from a group of variables accepted as defining the factor with each of the variables given equal weight (Gorsuch, 1983). In our item-level factor analysis, the multidimensional item response model is defined as

$$P_j(\underline{\theta}) = \int_{-\infty}^{\sum_{m=1}^{M} a_{g_m(j)m}\theta_m - b_j} \phi(t)\,dt \qquad (1)$$

where m is the m-th latent trait dimension (m=1, 2, ..., M), g is the g-th item group (g=1, 2, ..., G), and j is the j-th item (j=1, 2, ..., n). The total of n items are composed into G_m groups in each dimension m. The parameter, $a_{gm(j)m}$, is a discrimination power of the j-th item nested with the g-th group of the m-th dimension ($j \in g_m$ for each dimension m), b_j is an intercept parameter of the j-th item, and θ_m is a latent trait variable in the m-th dimension.

The following table is the illustration of a possible test structure:

item	dimension m					intercept
j	1	2	3	.	M	
1	g_1=1	g_2=1	g_3=1		g_M=1	
2						
3	g_1=2					
.		g_2=2			g_M=2	
.			g_3=2			b_j
.		.				
.	.	.	.			
.	.	.	.		.	
.	.	g_2=G_2	.		.	
.	g_1=G_1		.		g_M=G_M	
n			g_3=G_3			

EM algorithm for the parameter estimation

If we denote the item response as

W_{ji}=1 if person i responds to item j correctly

and

W_{ji}=0 if person i responds to item j incorrectly,

then the conditional probability of the l-th binary response pattern, (W_j), is

$$L_l(\underline{\theta}) = P_j((W_j)|\underline{\theta})$$
$$= \prod_{j=1}^{n} [P_j(\underline{\theta})]^{W_j}[1 - P_j(\underline{\theta})]^{1-W_j} \quad (2)$$

The marginal probability of the l-th response pattern is given by being integrated by the prior distribution $g(\underline{\theta})$, that is,

$$\tilde{P}_l = P_j((W_j))$$
$$= \int_{\underline{\theta}} P_j((W_j)|\underline{\theta})\, g(\underline{\theta})\, d\underline{\theta} \quad (3)$$
$$= \int_{\underline{\theta}} L_l(\underline{\theta})\, g(\underline{\theta})\, d\underline{\theta}$$

Given the frequencies, r_l, of the l-th response pattern for n items in a sample of N persons, the number of distinct patterns is $s \leq \min(2^n, N)$, and the probability of the sample is

$$\mathcal{L} = \frac{N!}{\prod_{l=1}^{s} r_l!} \prod_{l=1}^{s} \tilde{P}_l^{r_l} \quad (4)$$

The first derivative of the log-likelihood of Equation 4 is

$$\frac{\partial \ln \mathcal{L}}{\partial a_{g_m(j)m}} = \frac{\partial}{\partial a_{g_m(j)m}} \sum_{l=1}^{s} r_l \ln \tilde{P}_l$$
$$= \sum_{l=1}^{s} \frac{r_l}{\tilde{P}_l} \frac{\partial \tilde{P}_l}{\partial a_{g_m(j)m}} \quad (5)$$
$$= \int_{\underline{\theta}} \sum_{l=1}^{s} \frac{r_l}{\tilde{P}_l} \sum_{j \in g_m} \frac{L_l(\underline{\theta})}{[P_j(\underline{\theta})]^{W_j}[1-P_j(\underline{\theta})]^{1-W_j}}$$
$$\cdot \frac{\partial}{\partial a_{g_m(j)m}} [P_j(\underline{\theta})]^{W_j}[1-P_j(\underline{\theta})]^{1-W_j}\, g(\underline{\theta})\, d(\underline{\theta})$$

If $W_j = 1$, the second term of the Equation 5 becomes

$$\frac{L_l(\underline{\theta})}{P_j(\underline{\theta})} \frac{\partial P_j(\underline{\theta})}{\partial a_{g_m(j)m}} \quad (6)$$

and, if $W_j=0$, it becomes

$$-\frac{L_1(\underline{\theta})}{1 - P_j(\underline{\theta})} \frac{\partial P_j(\underline{\theta})}{\partial a_{g_m(j)m}} \tag{7}$$

Therefore, Equation 5 becomes

$$\sum_{j\in g_m} \int_{\underline{\theta}} \sum_{l=1}^{s} \frac{r_l}{\tilde{P}_l} \frac{W_j - P_j(\underline{\theta})}{P_j(\underline{\theta})\,[1-P_j(\underline{\theta})]} L_l(\underline{\theta}) \frac{\partial P_j(\underline{\theta})}{\partial a_{g_m(j)m}} g(\underline{\theta})\, d\underline{\theta} \tag{8}$$

$$= \sum_{j\in g_m} \int_{\underline{\theta}} \frac{\bar{r}_j - \bar{N} P_j(\underline{\theta})}{P_j(\underline{\theta})\,[1-P_j(\underline{\theta})]} \frac{\partial P_j(\underline{\theta})}{\partial a_{g_m(j)m}} g(\underline{\theta})\, d\underline{\theta}$$

where

$$\bar{r}_j = \sum_{l=1}^{s} \frac{r_l W_j L_l(\underline{\theta})}{\tilde{P}_l} \tag{9}$$

and

$$\bar{N} = \sum_{l=1}^{s} \frac{r_l L_l(\underline{\theta})}{\tilde{P}_l} \tag{10}$$

The multiple integral in this equation may be evaluated numerically by repeated Gauss-Hermite quadrature as follows:

$$\sum_{j\in g_m} \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{r}_{j,q_1,q_2,\ldots,q_M} - \bar{N}_{q_1,q_2,\ldots,q_M} P_j(X)}{P_j(X)\,[1-P_j(X)]} \frac{\partial P_j(X)}{\partial a_{g_m(j)m}} A(X_{q_1}) A(X_{q_2}) \cdots A(X_{q_M}) \tag{11}$$

The expected frequency

$$\bar{r}_{j,q_1q_2\ldots q_M}$$

is an entry in a Q^M dimensional array in which each cell corresponds to an m-tuple of quadrature points for a given item. The entries in this table are the numbers of examinees with abilities equal to the vector $\underline{X}$ who are expected to respond correctly to the item, given the sample data. The quantity

$$\bar{N}_{q_1q_2\ldots q_m}$$

is the margin of this array summed over items; it is the expected number of

persons with ability $\underline{X}$ and is normalized to the sample size.

After obtaining the expected frequencies, the following normal equations are iteratively solved:

$$\begin{bmatrix} a_{g_m} \\ b_j \end{bmatrix}_{h+1} = \begin{bmatrix} a_{g_m} \\ b_j \end{bmatrix}_h + \begin{bmatrix} t_{g_m} \\ t_j \end{bmatrix} \begin{bmatrix} V_{g_m g_m} & V_{g_m b_j} \\ V_{g_m b_j} & V_{b_j b_j} \end{bmatrix}^{-1} \tag{12}$$

where

$$t_{g_m} = \sum_{j \in g_m} \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{r}_{j,q_1,q_2,\ldots,q_M} - \bar{N}_{q_1,q_2,\ldots,q_M} P_j(X)}{P_j(X)\,[1-P_j(X)]} \frac{\partial P_j(X)}{\partial a_{g_m(j)m}} A(X_{q_1}) A(X_{q_2}) \ldots A(X_{q_M}) \tag{13}$$

$$t_{b_j} = \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{r}_{j,q_1,q_2,\ldots,q_M} - \bar{N}_{q_1,q_2,\ldots,q_M} P_j(X)}{P_j(X)\,[1-P_j(X)]} \frac{\partial P_j(X)}{\partial b_j} A(X_{q_1}) A(X_{q_2}) \ldots A(X_{q_M}) \tag{14}$$

for any j and j' ($j \in g_m$ and $j' \in g_m$),

$$V_{g_m g_{m'}} = \sum_{j \in g_m,\, j \in g_{m'}} \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{N}_{q_1,q_2,\ldots,q_M}}{P_j(X)\,[1-P_j(X)]} \frac{\partial P_j(X)}{\partial a_{g_m(j)m}} \frac{\partial P_j(X)}{\partial a_{g_{m'}(j)m'}} A(X_{q_1}) A(X_{q_2}) \ldots A(X_{q_M}) \tag{15}$$

for any j, $g \neq g'$, $m \neq m'$,

$$V_{g_m b_j} = \sum_{j \in g_m,\, j \in g_{m'}} \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{N}_{q_1,q_2,\ldots,q_M}}{P_j(X)\,[1-P_j(X)]} \frac{\partial P_j(X)}{\partial a_{g_m(j)m}} \frac{\partial P_j(X)}{\partial b_j} A(X_{q_1}) A(X_{q_2}) \ldots A(X_{q_M}) \tag{16}$$

and finally

$$V_{b_j b_j} = \sum_{q_M=1}^{Q} \cdots \sum_{q_2=1}^{Q} \sum_{q_1=1}^{Q} \frac{\bar{N}_{q_1,q_2,\ldots,q_M}}{P_j(X)\,[1-P_j(X)]} \left[\frac{\partial P_j(X)}{\partial b_j}\right]^2 A(X_{q_1}) A(X_{q_2}) \ldots A(X_{q_M}) \tag{17}$$

Simulation Study

The estimation process of the group-factors were implemented in the TESTFACT (Wilson, England, Wood, and Gibbons, 1987) program. The RESGEN simulation program (Muraki, 1991) was used to generate a multidimensional dichotomous item responses based on N(0,1). The number of simulees is 5000. In Table 1, a set of item parameters used for the simulation are presented.

****
Insert Table 1 here
****

From item response parameters, we can compute the factor loading, α_m, and threshold, γ_j, by

$$\alpha_m = \frac{a_m}{\sqrt{d_j}} \tag{18}$$

and

$$\gamma_j = -\frac{b_j}{\sqrt{d_j}} \tag{19}$$

where

$$d_j = 1 + \sum_{m=1}^{M} a_m^2$$

In Table 2, the factor loadings and thresholds computed from those item parameters are presented.

****
Insert Table 2 here
****

For the first solution (Solution 1), the item parameters were estimated without grouping items. Second (Solution 2), four item-groups were constructed, such that

Group	Item
1	1, 2
2	3, 4, 5
3	6, 7, 8
4	9, 10

For the third solution (Solution 3), the item grouping was altered for each dimension, such that

Dimension	Group	Item
1 and 2	1	1, 2, 3, 4, 5
	2	6, 7, 8, 9, 10
3 and 4	1	1, 2, 6, 7, 8
	2	3, 4, 5, 9, 10

For each solution, the convergence criterion was set at 0.01, and three quadrature points were used for integration. Estimated slope parameters were converted to the factor loadings. The principal factor loadings were, then,

rotated by the varimax method. The varimax factor loadings for three solutions are presented in Tables 3, 4, and 5, respectively. The -2 log-likelihood for each solution is presented in Table 6. Since the rank of our original factor structure is three, the first three factors appear to be interpretable. Four sets of items can be clearly identifiable by the factor structure. If we impose this structure to the factor model (Solution 2), we obtain the best fit, as shown in Table 6.

****
Insert Tables 3, 4, 5, and 6 here
****

Analysis of the LSAT Section 7

Bock, Gibbons, and Muraki (1988) showed that the data of LSAT Section 7 have two factors. The varimax factor loadings of the two-factor solution (Solution 1) are shown in Table 7. If we set a_{21}, a_{23}, a_{24}, and a_{25} zero and estimated other slopes, the difference of chi-square fit between this solution and Solution 1 is 8.93 (Df=4). If we set a_{21}, a_{24}, and a_{25} zero and estimated other slopes, the difference of chi-square fit between this solution (Solution 2) and Solution 1 becomes smaller (χ^2=0.51, Df=3). The estimated varimax factor loadings in Solution 2 are presented in Table 8. Since the factor axes rotated to the varimax criteria show that items 2 and 3 determines the second factor. We grouped these two items and obtained the two factor solution (Solution 3). The promax group-factor loadings of this solution are presented in Table 9. The difference of chi-square fit between Solution 1 and Solution 3 is 6.38 (Df=8). The correlation between the first and second promax group-factors is 0.651. Thus, we accomplished a parsimonious description of the factor structure of the LSAT data.

****
Insert Tables 7, 8, and 9 here
****

Discussion

The group-factor analysis is a special case of the confirmatory factor analysis. As seen in the normal equations 13, 15, and 16, the group factors can be computed based on the collapsed gradient and Hessian matrix by summing up the elements according to the item group structure. Unlike the method of summing up the dichotomous item responses and analyzing the summed responses as continuous variables, this full-information method can control the item difficulty and guessing of each item. The comparison of these two methods in various conditions should be investigated.

This study is the first attempt to use Bock and Aitkin's (1981) item level factor analysis procedure in a confirmatory way. More optional features will be added to the TESTFACT program upon the result of this study. Since each of achievement tests is generally constructed based on the careful test design, confirmatory analyses incorporating flexible assumptions, including the group-level factor structure, will be appropriate process to analyze test data.

References

Bock, R. D., & Aitkin. M. (1981). Marginal maximum likelihood estimation of item parameters: Application of an EM algorithm. Psychometrika,46, 443-458.

Bock, R. D., Gibbons, R., & Muraki, E. (1988). Full-information factor analysis. Applied Psychological Measurement, 12(3), 261-280.

Muraki, E. (1991). RESGEN: Item Response Generator (in preparation). N.J.: Educational Testing Service.

Table 1

Original Intercept and Slope Parameters

Simulation

Item	Intercept	a_1	a_2	a_3	a_4
1	1.5	2.0	0.0	0.6	1.8
2	1.3	2.0	0.0	0.6	1.8
3	-1.5	2.0	0.0	1.8	0.6
4	-1.3	2.0	0.0	1.8	0.6
5	-1.5	2.0	0.0	1.8	0.6
6	-1.5	0.0	2.0	0.6	1.8
7	-1.3	0.0	2.0	0.6	1.8
8	-1.5	0.0	2.0	0.6	1.8
9	1.3	0.0	2.0	1.8	0.6
10	1.5	0.0	2.0	1.8	0.6

Table 2

Original Thresholds and Factor Loadings

Simulation

Item	Threshold	α_1	α_2	α_3	α_4
1	-0.51	0.68	0.00	0.21	0.61
2	-0.44	0.68	0.00	0.21	0.61
3	0.51	0.68	0.00	0.61	0.21
4	0.44	0.68	0.00	0.61	0.21
5	0.51	0.68	0.00	0.61	0.21
6	0.51	0.00	0.68	0.21	0.61
7	0.44	0.00	0.68	0.21	0.61
8	0.51	0.00	0.68	0.21	0.61
9	-0.44	0.00	0.68	0.61	0.21
10	-0.51	0.00	0.68	0.61	0.21

Table 3

Estimated Varimax Factor Loadings

Simulation Solution 1

Item	α_1	α_2	α_3	α_4
1	0.664	0.199	0.583	0.022
2	0.667	0.190	0.591	-0.009
3	0.905	0.143	0.084	-0.085
4	0.915	0.130	0.040	0.063
5	0.892	0.169	0.075	0.003
6	0.084	0.879	0.245	0.010
7	0.087	0.875	0.235	0.060
8	0.074	0.865	0.257	0.020
9	0.318	0.811	-0.261	-0.056
10	0.309	0.825	-0.280	-0.078

Table 4

Estimated Varimax Factor Loadings

Simulation Solution 2

Item	α_1	α_2	α_3	α_4
1	0.260	0.795	0.305	0.045
2	0.260	0.795	0.305	0.045
3	0.124	0.896	-0.162	-0.018
4	0.124	0.896	-0.162	-0.018
5	0.124	0.896	-0.162	-0.018
6	0.904	0.138	0.101	-0.002
7	0.904	0.138	0.101	-0.002
8	0.904	0.138	0.101	-0.002
9	0.783	0.246	-0.389	0.010
10	0.783	0.246	-0.389	0.010

Table 5

Estimated Varimax Factor Loadings

Simulation Solution 3

Item	α_1	α_2	α_3	α_4
1	0.845	0.238	0.264	0.000
2	0.845	0.238	0.264	0.000
3	0.886	0.142	-0.210	0.000
4	0.886	0.142	-0.210	0.000
5	0.886	0.142	-0.210	0.000
6	0.158	0.892	0.055	0.000
7	0.158	0.892	0.055	0.000
8	0.158	0.892	0.055	0.000
9	0.219	0.771	-0.437	0.000
10	0.219	0.771	-0.437	0.000

Table 6

-2 Log likelihoods

Simulation

Solution	-2 Log likelihood
1	43430
2	43416
3	43446

Table 7

Estimated Varimax Factor Loadings

LSAT-7 Solution 1

Item	α_1	α_2
1	0.560	0.168
2	0.161	0.776
3	0.422	0.465
4	0.411	0.184
5	0.453	0.097

Table 8

Estimated Varimax Factor Loadings

LSAT-7 Solution 2

Item	α_1	α_2
1	0.542	0.199
2	0.179	0.674
3	0.387	0.536
4	0.449	0.164
5	0.399	0.146

Table 9

Estimated Promax Factor Loadings

LSAT-7 Solution 3

Item	α_1	α_2
1	0.006	0.502
2	0.626	0.034
3	0.626	0.034
4	0.006	0.502
5	0.006	0.502